NOISE AND PERTURBATIONS

Ch.1
Fundamental notions

Study of fluctuations using signal theory

This method has a more physical character and is preferred by all those who are interested in the measurement aspect, in which case the fluctuations are identified with random signals.

Signals

- Periodic signals

$$v(t) = v(t+T) \qquad (1.1)$$

- Aperiodic signal

The signal represents the materialization of the energy propagation in a circuit.
There are two signal families:
1) Periodic signals, characterized by a waveform that is repeated after a fixed interval, called the period and noted with T, Eq. (1.1).
2)Aperiodic signals, characterized by the fact that Eq. (1.1) is not satisfied for any value of T. This is the case of "vocal" type signals, transient signals and random signals.
We are interested in this course especially to the random signals, which are characterized by a total uncertainty of their evolution. Examples of random signals are: cosmic, atmospheric disturbances or parasites produced by industrial activity.
To characterize a signal in the time domain, we call the following parametric: 1) the mean value, 2) the mean square value, 3) the form factor, 4) the ridge factor.

The average value

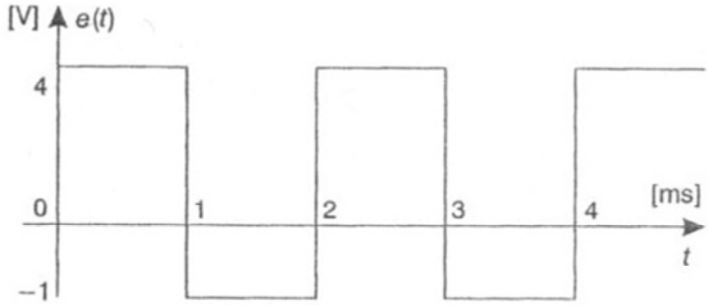


Fig.1.1

$$I_0 = \frac{1}{T}\int_0^T i(t)\,dt \qquad (1.2)$$

Definition

The average value is the ratio between the area defined in the waveform during a period and the value of that period.

For a signal i (t), the mean value I0 is given by Eq. (1.2), where T is the observation time.

If the signal is not periodic, Eq. (1.2) is applied by identifying the period T with the observation interval (as high as possible).

Example

$$E_0 = \frac{1}{(2ms)}\left((4V)(1ms) + (-1V)(1ms)\right) = 1.5V$$

In the case of the signal in Fig.1.1, we can calculate the average value as defined.

The mean square value

Effective value

$$\underbrace{\frac{1}{R}\frac{1}{T}V^2\int_0^T \cos^2 \omega t dt}_{\text{signal power}} = \underbrace{\frac{1}{R}V_{CC}^2}_{\text{Power in DC}} \qquad (1.3)$$

$$V_{ef} = V_{CC} = V/\sqrt{2} \qquad (1.4)$$

The mean square value

$$V_{qm} = \left(V_{ef}\right)^2 = \frac{1}{T}\int_0^T v^2(t)\,dt \qquad (1.5)$$

We consider a periodic signal v (t) = V cos (ωt). The effective value is noted with Vef, and its mean square value with Vqm.

Effective value

This is a measurable size, which, traditionally, is introduced with the help of the dissipated power in a load resistance.
By definition, the effective value of a signal is equal to the value of DC that would dissipate, in the same charge, the same power as the studied signal.
Thus, for the harmonic signal v (t) considered above, we can write Eq. (1.3), whence Eq. (1.4).
In conclusion, the effective value of a signal gives us the possibility to calculate the power it dissipates in a charge, without being interested in the waveform of the signal.

The mean square value

By definition, the mean squared value is equal to the square of the effective value. Eq. (1.5)

Calculation of the mean square value

- Raise the original signal to the square
- Calculate the area under this curve for a period
- Divide the area by the period

Example

$$E_{qm} = \frac{(4V)^2(1ms) + (-1V)^2(1ms)}{(2ms)} = 8.5\,V^2$$

$$E_{ef} = \sqrt{8.5} = 2.91\,V$$

$$P = E_{msq}/R \qquad (1.6)$$

For the waveform in Fig. (1.1), the mean squared value is calculated as in the example. The effective value is the square root of the quadratic mean value. We can conclude that a DC voltage of 2.91V dissipates in a resistor R the same amount of heat as the signal in Fig. (1.1).

With the help of the relation (1.5) we have the relation (1.6).
Eq. (1.6) offers the possibility to regard the mean square value as the power dissipated by the signal in a 1Ω load. This power is called *normalized power.*

In the case of noise, the mean squared value is the first significant parameter, because the average value of the noise is zero.

The Correlation

$$v_a = A\sin(\omega t) \qquad (1.7a)$$
$$v_b = B\sin(\omega t + \phi) \quad (1.7b)$$
$$v_b = \underbrace{(B\cos\phi)\sin(\omega t)}_{\text{term 1}} + \underbrace{(B\sin\phi)\cos(\omega t)}_{\text{term 2}} \quad (1.7c)$$

Definition
Two waveforms are coherent if their temporal evolution is similar, except for amplitudes that can be distinguished by a scale factor.

Two coherent signals are called *totally correlated*. In this case:
1) The functions that describe their evolution over time are identical;
2) The gap between the two waveforms is null;
3) Their amplitudes are not necessarily equal.

If the phase shift is not zero, but remains weak enough, then the signals are *partially correlated*. In this case, the danger to be avoided is to have a too important phase shift, which could radically change the behavior. For example, a phase shift of π / 2 imposed on a sinusoid transforms it into a cosineusoid.

Example
Let us consider the elementary case of two sinusoidal signals, of the same frequency, but of different amplitudes, offset with a weak angle ø (so partially correlated signals), Eqs. (1.7).
Developing, the signal vb can be put in equivalent form (1.7c).
Term 1 represents a signal totally correlated with you (same function, no phase

shift), while term 2 is decorated with respect to you (another time function). Depending on the value of ø, the amplitudes of these components can be changed, even canceled.

This example introduces a technique often used in noise theory: when we have two noise generators na and nb partially correlated, na can be decomposed into two generators na1 and na2, so that na1 is totally correlated with nb, and na2 independently with respect to nb (the same decomposition can be applied to signal nb).

The case of fluctuations

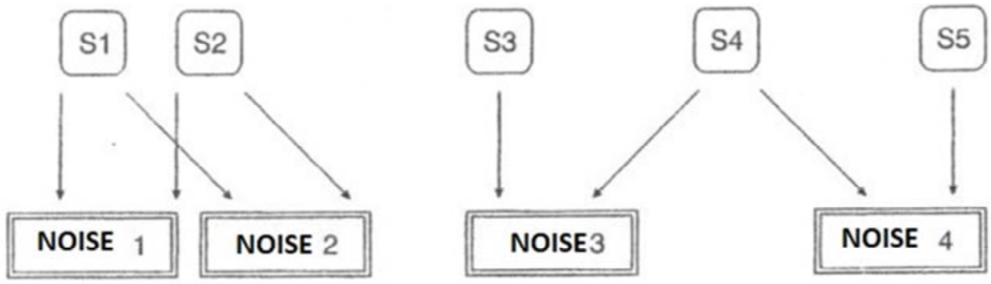


Fig.1.2

Random signals (noise) are not described by analytical expressions in relation to time, but in probabilistic terms. The only "link" between the evolution of two fluctuations is possible only due to possible common origins.
For example, if a noise signal is produced by a certain physical phenomenon, which is at least partially responsible for generating a second noise signal, it is obvious that the two noise signals cannot evolve independently of one another and are therefore , partially related. All the possible situations are illustrated in Fig.1.2., Where the sources are noted S1, S2, S3, S4 and S5. These sources symbolize either physical phenomena that are at the origin of the noise, or equivalent generators introduced for modelling purposes.
Noise signals 1 and 2 are totally correlated, 3 and 4 are partially correlated, but 1 and 3 (or 2 and 4, or 1 and 4, or 2 and 3) are independent.

Correlation coefficient (classical approximation)

$$\overline{v^2} = \overline{\left(v_a + v_b\right)^2} = \overline{v_a^2} + 2\overline{v_a v_b} + \overline{v_b^2} \qquad (1.8a)$$

$$\overline{v^2} = \frac{1}{T}\int_0^T \left(A^2 \sin^2 \omega t + 2AB\sin(\omega t)\sin(\omega t + \phi) + B^2 \sin^2(\omega t + \phi)\right) dt \qquad (1.8b)$$

$$\overline{v^2} = \frac{A^2}{2} + AB\cos\phi + \frac{B^2}{2} \qquad (1.8c)$$

We need a measure to appreciate the correlation between two signals. We resume the case of the two deterministic signals (1.7a) and (1.7b) and suppose that they are applied simultaneously to the terminals of a unit resistance. We are interested in the power dissipated in this resistance.

As we have seen, we need the mean squared value of the total voltage at the load terminals, Eq. (1.8a). Development is possible because the average value of the sum is the sum of the average values. For the product, its average value is equal to the product of the average values if and only if the signals are independent. Substituting relations (1.7a) and (1.7b) into Eq. (1.8a), we have (1.8b). After calculations, we reach (1.8c)

Correlation coefficient (classical approximation)

$$\overline{v^2} = v_{ef}^2 = \left(\frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}} \right)^2 \qquad (1.8d)$$

$$\overline{v^2} = \frac{A^2}{2} + \frac{B^2}{2} \qquad (1.8e)$$

$$c = \frac{\overline{v_a v_b}}{\sqrt{\overline{v_a^2}\,\overline{v_b^2}}} \quad (1.9) \qquad c = \frac{\overline{v_a v_b}}{(v_a)_{ef}(v_b)_{ef}} = \frac{AB\cos\phi}{AB} = \cos\phi \quad (1.10)$$

We have the following particular cases:
1) If ø = 0, the two waveforms are totally correlated. Eq. (1.8c) becomes (1.8d), which shows that for two totally correlated signals, the effective value of the total voltage is the sum of the effective values of the two signals.
2) We consider the case ø = π / 2, which corresponds to two independent signals. In this case, the normalized power dissipated in the load is the sum of the individual normalized powers, Eq. (1.8e).
3) If 0 <ø <π / 2, we are in the intermediate situation, ie partial correlation. The correlation coefficient is defined by the expression (1.9), which leads to Eq. (1.10).

Equality (1.10) proves that the absolute value of the correlation coefficient is between 0 (for independent signals) and 1 (for totally correlated signals). The possible value -1 indicates a total correlation between two signals whose waveforms are in phase opposition and subtract from each other.

Study of fluctuations using probability theory

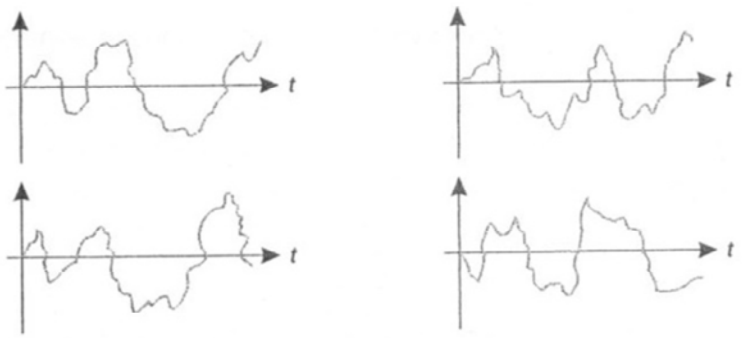


Fig.1.3

Any physical quantity has fluctuations around its equilibrium average, which can be considered as a random process. In the case of most physical phenomena, we admit that the instantaneous values of the fluctuations are distributed in a Gaussian manner around the average and in many cases we have strong arguments (based on the central limit theorem) in favor of this hypothesis.

Random variable
In electronic circuits, fluctuations of voltages and currents are described using the concept of random variable. By definition, a random variable is a correspondence that allows us to assign numerical values to the results of a statistical experience (test). We should note that the notion of a random variable is a misleading name, because it is not a variable in the classical sense of the term, but more of a random function.

A random variable can be continuous (if the values it belongs to a continuous interval) or discrete (when we consider only a few well-defined values). For example, when trying to throw a coin, we can assign the value +1 to the appearance of a girl and the value -1 when the other face appears. In this case the random variable is discrete and takes only two values. Similarly, the fluctuation of the total number of charge carriers in a semiconductor leads to considering a discrete random variable

having a huge number of values.
If the test considered consists of throwing an arrow at the target, we can define as the random variable the distance between its point of impact and the center of the target. This time, the variable can take any value and so we are in the presence of a continuous random variable. Thus, if the only property retained to characterize a noise current is the instantaneous value of its amplitude, this leads us to consider a continuous random variable.

Random process (stochastic)
In the case of a random process, the result of the test is not a number, but a random function of time. All random functions resulting from a series of tests are called a set.
For example, the thermal noise of a resistor is a random process. The evolution of the noise voltage of this resistor defines a random variable (sample) and the totality of the noise voltages generated by many similar resistors constitutes the set, Fig. 1.3.

Study of fluctuations using probability theory

- **Probability density**

$$f(x) = \lim_{\substack{\Delta x \to 0 \\ N \to \infty}} \frac{(\text{The number of values in the range } \Delta\text{x located at x}) \cdot \Delta x}{\text{The total number of values N}}$$

- **The distribution function**

$$F(x) = \int_{-\infty}^{x} f(y)\,dy \qquad (1.11)$$

Probability density
In the case of noise, we can adopt as a fluctuating magnitude the current (or voltage) of noise. We consider an ideal situation, when we have the possibility to measure at any moment this size and be x a value of its amplitude that we will represent on the axis of the abscissae. If we divide this axis into elementary interval Δx and count the number of measured values that belong to each interval, we can approximate the probability density f9x0 by the expression on the slide.
Then the probability of having the amplitude of the current (or voltage) of noise within the elementary range dx centered around x will be equal to f (x) dx.

The distribution function
It is defined as the probability that the measured amplitude is less than a given value x. It is calculated with Eq. (1.11).

Characterization of fluctuating quantities

- Ergodism
- Stationarity
- averges

$$m_n(x) = \overline{x^n(t)} = \int_{-\infty}^{+\infty} x^n f(x)\,dx \quad (1.12)$$

$$m_2(x-\bar{x}) = \text{var}.\{x\} = \overline{(x-\bar{x})^2} = \overline{x^2} - (\bar{x})^2 \quad (1.13)$$

$$\sigma_x = \sqrt{\text{var}.(x)} \quad (1.14)$$

An important property of all fluctuations is the impossibility of specifying its amplitude at each moment of time; for this reason, we are forced to adopt a statistical description.

In general, a fluctuating quantity is completely characterized by its probability density (or its distribution function). In the case of electric noise, this is rarely possible and we must adopt an incomplete description, using the average.

There are two possibilities to calculate the averages of the amplitude of a current (or a voltage) of noise. The first approach is based on revealing the instantaneous values of this current, over a long period of time, followed by the calculation of the average. Thus we obtain the <u>*temporal average*</u> of the current i (t), which will be denoted ¯i. This is the case found in noise measurements.
The second approach is to consider a fictitious set of many identical systems, all delivering the same noise current i (t). If we are in the ideal case where we can simultaneously observe all the instantaneous values at a given time of time t_0, then it will be possible to calculate their average $\langle i \rangle$, which is called <u>*statistical average*</u>. This way of estimating the averages is used mainly in the calculation of noise.

<u>Ergodism</u>

A random process is called ergodic if the temporal average is equal to the statistical average. Electrical noise is an ergodic process.

Stationarity

A process is stationary if all averages are time independent. The most important types of noise can be described by stationary processes. It should be noted that any ergodic process is also stationary, but the reciprocal is not valid.

Averages

The formula for calculating the average of order n, of a random variable x assumed to be continuous, having the probability density f(x), is Eq. (1.12).
The most important averages that characterize a fluctuation x (t) are the first-order and second-order averages (which is the average quadratic value). In the case of a noise current, the average value is always zero, the first non-zero being ¯ (i ^ 2).
If the average value is not zero, the centered variable (x − ¯x) is introduced, whose average of order two is called the variance, Eq. (1.13). The quantity (1.14) , called *standard* deviation, is also used.

The physical meaning associated with the various average values

1) The first order average represents a component of continuous current.
2) The square of the first order average can be identified with the continuous power developed in a resistance of 1Ω.
3) The average of order two represents the total power dissipated in a resistance of 1Ω.
4) The variance is the power of the signal component dissipated in the 1Ω resistor.
5) The standard deviation represents the efficace value of the signal component of the current.

If x (t) represents a current or voltage, then in the case of an ergodic process we can interpret the first and second order averages as follows:
1) The first order average represents a component of continuous current.
2) The square of the first order average can be identified with the continuous power developed in a resistance of 1Ω.
3) The average of order two represents the total power dissipated in a resistance of 1Ω.
4) The variance is the power of the signal component dissipated in the 1Ω resistor.
5) The standard deviation represents the efficace value of the signal component of the current.

Central limit theorem

$$Y = \sum_{i}^{n} X_i$$

$$Y \xrightarrow{n\to\infty} \text{Normal variable}$$

$$\langle Y \rangle = n \langle X_i \rangle \text{ , var.}(Y) = n\sigma^2$$

Normal variable: $$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{\left(x-\bar{x}\right)^2}{2\sigma^2} \right\}$$

We note with X1, X2, ..., Xn, random variables having the same probability density (and therefore the same mean value and the same variance). If n tends to infinity, the sum of random variables tends to a normal law.

Characterization using two variables

- Density of probability f(x,y)
- Distribution function F(x,y)

$$F\left(x,y\right)=\int_{-\infty}^{x}\int_{-\infty}^{y}f(u,v)dudv \quad \left(1.15\right)$$

The disadvantage of the one-dimensional description lies in the inability to access the power spectrum of the studied fluctuation. Characterization by means of a single random variable allows only the continuous component to be separated from the rest of the signal, by means of the 1st and 2nd order media.
The next step is to consider the statistics of the couple of values of the current (or voltage), separated by well defined time intervals. Thus we get to consider two random variables.

Probability density f (x, y)
f (x, y) is the function which, multiplied by the elementary area dxdy, gives us the probability of having simultaneously the first variable between x and x + dx and the second between y and y + dy.

Distribution function F (x, y)
It is defined as the probability of having the variables with values lower than two predetermined values x and y, Eq. (1.15).

Characterization using two variables

- Averages

$$m_{ik}(x,y) = \overline{x^i y^k} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^i y^k f(x,y)dxdy \quad (1.16)$$

$$m_{00}(x,y) = 1 \quad (1.17a)$$

$$m_{01} = \overline{y} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} yf(x,y)dxdy \quad (1.17b)$$

$$m_{10} = \overline{x} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xf(x,y)dxdy \quad (1.17c)$$

In the case of electrical noise, $\overline{x} = \overline{y}$ = 0 and the first significant averages remain $\overline{x^2}$, $\overline{y^2}$ si $\overline{xy}$.

Characterization using two variables

- Centred variables

$$\mu_{ik}(x,y)=\overline{\left(x-\overline{x}\right)^{i}\left(y-\overline{y}\right)^{k}}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(x-\overline{x}\right)^{i}\left(y-\overline{y}\right)^{k} f(x,y)dxdy \quad (1.18)$$

$$\mu_{02}=\overline{\left(y-\overline{y}\right)^{2}}$$

$$\mu_{20}=\overline{\left(x-\overline{x}\right)^{2}}$$

Covariance

$$\mu_{11} = \overline{\left(x - \overline{x}\right)\left(y - \overline{y}\right)} = \overline{xy} - \overline{x}\,\overline{y} \quad (1.19)$$

If the two fluctuating quantities x and y are independent, (ie a value taken by x does not influence the values taken by y), then the covariance is null and the product average is equal to the product of the averages.

Covariance allows the definition of the power spectrum (spectral density) of a random process.

Correlation

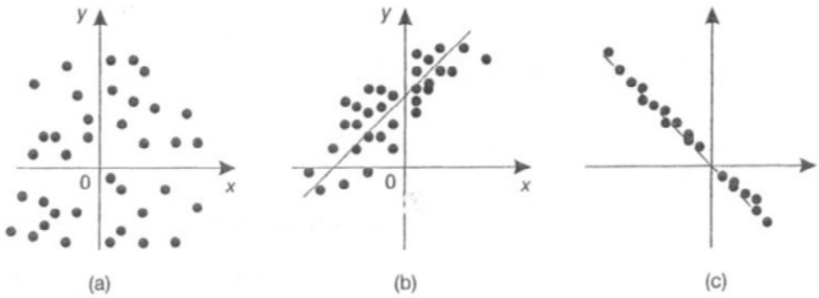


Fig.1.4

$$c = \frac{\mu_{11}}{\sigma_x \sigma_y} \quad (1.20) \qquad -1 \le c \le +1$$

The correlation is the relationship between two fluctuating quantities x and y, which tend to influence each other, in a way that cannot be uniquely explained by the theory of hazard.

An intuitive method of observing whether or not the quantities x and y are correlated is to represent in a plane all pairs of values (x, y) obtained, Fig.1.4.
If the two quantities are not correlated, we can expect these points to be evenly distributed throughout the plane, Fig. 1.4a.
If the two quantities depend on each other, then the points are distributed around the curve that describes their functional dependence, Fig. 1.4b.
The total correlation is illustrated in Fig. 1.4c.

Correlation coefficient
By definition, the *correlation coefficient* between two fluctuating quantities x and y is the ratio between their covariance and the product of their standard deviations, Eq. (1.20).
Depending on the values of the correlation coefficient, we can establish the following classification:

1. If c = 0, then x and y are independent
2. If c = 1, then x and y are totally correlated (linear dependence between x and y)

3. If $|c| \leq 1$, then the quantities x and y are partially correlated.
Except for nonlinear circuits, the correlation between the noise quantitiesis always linear.

Partial correlation

$$y = ax + z, \quad a = \text{ct.} \quad (1.21a)$$

unde

$$\overline{x} = \overline{y} = \overline{z} \quad (1.21b)$$

$$c = (sign\ a)\left(1 + \overline{x^2} / \overline{a^2 x^2}\right)^{-1/2} \quad (1.22)$$

In the case of two fluctuating quantities x and y, partially correlated, we have the possibility to express y as a sum of two terms, the first being totally correlated with x and the second (z) being independent, Eq.1.21a.
This decomposition is arbitrary and is not unique, so no physical meaning can be attached to it.

Autocorrelation function

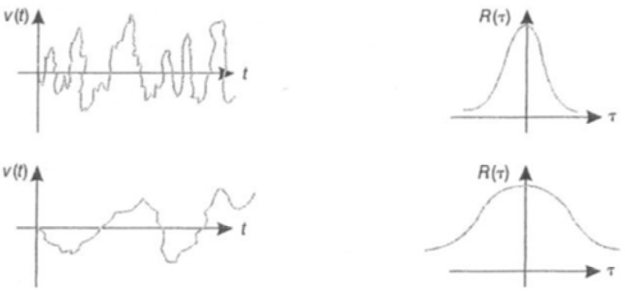


Fig.1.5

$$R(\tau) = \overline{v(t_1)\, v(t_1 + \tau)} \quad (1.23)$$

$$R(0) = \overline{v^2} \quad (1.24)$$

It is often important to know how quickly a random function changes Ii relation to time. This information is provided by the autocorrelation function.
If we make two measurements on the same noise source, at two different time points: x = v (t1) and y = v (t1 + τ).
If the noise v (t) is stationary, then the statistics do not depend on the time t1, but only on τ.

By definition, the autocorrelation function is the average of the product v (t1) and v (t1 + τ), Eq. (1.23).
According to Fig. 1.5, the width of the autocorrelation function is inversely proportional to the speed of variation of the random function.

R (τ) represents a measure of the correlation that exists between two values of the fluctuating quantity v (t), at two different time points. We observe that if in Eq. (1.23), v (t) is a periodic signal, then the knowledge of the signal at time t is sufficient to predict its evolution at time (t + τ); we say that the periodic signals are totally correlated with themselves. On the contrary, if v (t) is a fluctuating quantity, the correlation is very weak, because the knowledge of the signal at time t1, provides little information on its immediate evolution.

It should be noted that if $\tau = 0$, then the autocorrelation function is identified with the mean quadratic value, Eq. (1.24) and, given the physical meaning of the averages, the autocorrelation function is therefore equal to the total power developed by v (t) in a unitary rezistance.

In conclusion, the autocorrelation function and signal strength are related.

Energy and power spectra

- **Fourier transform**

$$F(f)=\int_{-\infty}^{+\infty} f(t)\exp(-j2\pi ft)\,dt \quad (1.25a)$$

$$f(t)=\int_{-\infty}^{+\infty} F(f)\exp(j2\pi ft)\,df \quad (1.25b)$$

The representation in the frequency domain of the time function f (t) is obtained with the help of the Fourier transform, Eq. (1.25a), and the inverse Fourier transform, (1.25b).

Spectrum of an aperiodic signal

$$S_f(V) = \frac{V_{ef}^2}{\Delta f} = \left(\frac{V_{ef}}{\sqrt{\Delta f}} \right)^2 \quad (1.26)$$

- Spectral density
- Normalized power

In the case of an aperiodic signal, the amplitude spectrum F(f), as well as the phase spectrum <F (f), are both continuous.

Spectral density

The power spectral density is defined as a real , even and positive function, Sf, which describes the distribution of the average power according to the frequency. Integrating throughout the frequency range, it leads to the normalized average total power (per ohm).

Thus, for a unit load and a signal v (t) of effective value Vef (assumed constant in the band Δf), we can write Eq. (1.26).

The quantity $V\_ef / \sqrt{\Delta f}$ is called the spectral density of the voltage; it is measured in $V / \sqrt{Hz}$. We can also introduce the spectral current density, measured in $A / \sqrt{Hz}$.

Normalized power

The product S_f (x) df represents the average power dissipated by the signal x (t) in a resistance of 1 Ω, in a frequency range df.

Parseval's theorem

$$\int_{-\infty}^{+\infty} x_1(t)\,x_2(t)\,dt = \int_{-\infty}^{+\infty} X_1^*(f)\,X_2(f)\,df \quad (1.27)$$

$$W = \int_{-\infty}^{+\infty} \overline{v^2(t)}dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(j\omega)F^*(j\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|F(j\omega)\right|^2 d\omega \quad (1.28)$$

This theorem establishes the transition between the temporal domain and the frequency domain for the product of two real functions, Eq. (1.27).

Using the symmetry property and considering a single signal, this theorem can be expressed in terms of signal energy, Eq. (1.28).

This equality, (1.28), proves that the total energy of a signal can be calculated in two different ways: 1) integrating the quantity | F ($j\omega$) | ^ 2 throughout the spectrum; 2) integrating the average quadratic value across the temporal domain. In this way an equivalence is established between the frequency spectrum and the average quadratic value of the signal.

Energy spectral density

$$S_{\omega}(W)=\frac{1}{2\pi}\left|F(j\omega)\right|^{2} \quad (1.29)$$

The spectral energy density represents how the energy of the aperiodic signal is distributed. Taking into account the relation (1.28), it is introduced by the relation (1.29).
By definition, the spectral energy density is a real, positive and even function, which, integrated throughout the all ω field, leads to the average total energy per ohm.
The energy spectral density can be changed by passing the signal through a quadripole.

Harmonic analysis of fluctuating quantities

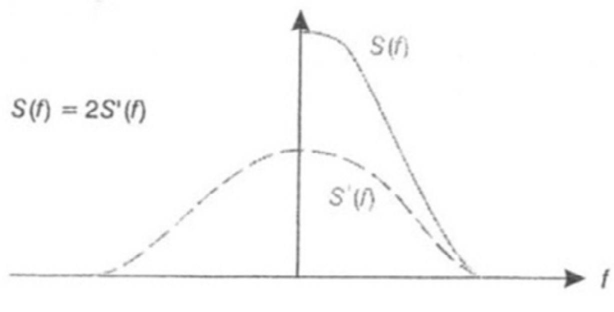


Fig.1.6

A fluctuating quantity x (t) is more assimilable with an aperiodic signal, which causes us to adopt the definitions (1.26) and (1.28) beforehand.
In signal theory, powers are defined on a domain ranging from −∞ to + ∞ ("bilateral" spectrum), while in noise theory a unilateral spectrum (from 0 to + ∞) is adopted.
The relationship between the bilateral spectral density (denoted S') and the unilateral spectral density (denoted S) is illustrated in Fig.1.6.

Wiener-Khintchine's Theorem

$$S_f(x) = \int_{-\infty}^{+\infty} R(\tau)\exp(-j\omega\tau)\,d\tau \quad (1.30)$$

$$R(\tau) = \int_{-\infty}^{+\infty} S_f \exp(+j\omega\tau)\,df \quad (1.31)$$

Let a fluctuation x (t) for which the autocorrelation function R (τ) exists; then the spectral power density Sf of the fluctuation x (t) is Fourier transform of the autocorrelation function R (τ) , Eq. (1.30), Eq. (1.31).

The superposition of several fluctuations

$$z(t) = x(t) + y(t)$$

$$\overline{z^2(t)} = \overline{\left(x(t) + y(t)\right)^2} = \overline{x^2(t)} + 2\overline{x(t)\,y(t)} + \overline{y^2(t)}$$

$$\overline{z^2(t)} = P_1 + 2P_{12} + P_2 \quad (1.32)$$

We want to calculate the power obtained by overlapping two fluctuations x (t) and y (t), so that z (t) = x (t) + y (t).
P1 and P2 represent the inherent powers of fluctuations and P12 is the cross power.
If the fluctuations are not correlated (when they come from two independent sources of noise), then P12 = 0 and in this case it turns out that the powers (and also the spectral densities and autocorrelation functions) have the additivity property.

The cross-correlation function
Spectral density of cross power

$$S_f\left(xy\right) = F\left\{R_{xy}\left(\tau\right)\right\} \quad (1.33)$$

The cross-correlation function is defined as the temporal mean of the product x (t) y (t + τ) and is denoted Rxy (τ). Two random processes for which Rxy (τ) = 0, whatever τ, are orthogonal. For example, if two processes are statistically independent and if at least one of them has a zero mean value, then they are orthogonal (but the reciprocal is not true).

Spectral cross power density is in Eq.(1.33)

The case of linear systems

$$S_f(y) = H(j2\pi f)H^*(j2\pi f)S_f(x) = \left|H(j2\pi f)\right|^2 S_f(x) \quad (1.34)$$

$$R_y(\tau) = \mathcal{F}^{-1}\left\{S_f(y)\right\} = \int_{-\infty}^{+\infty} \left|H(j2\pi f)\right|^2 S_f(x)\exp(j2\pi/\tau)\,df \quad (1.35)$$

$$R_{xy}(\tau) = h(\tau) \circ R_x(\tau) \quad (1.36)$$

$$S_f(xy) = H(j2\pi f)S_f(x) \quad (1.37)$$

$$S_f(yx) = S_f^*(xy) \quad (1.38)$$

Let consider a linear system with concentrated parameters, characterized by the transfer function H (jω), which is excited by a fluctuation x(t) of which we know the average and the spectral density Sf (x). In this case, the power spectral density of the fluctuation y(t) at the output is given in Eq. (1.34)
The output autocorrelation function will be transformed into inverse Fourier of Sf (y), Eq. (1.35)

1. It is shown that the cross-correlation function between the input x and the output y is identical to the convolution product between the autocorrelation function of the input and theimpulse response of the system, Eq. (1.36).
2. The spectral density of cross power is a complex number, calculated with relation (1.37).
3. From the above, we have the property (1.38).

Conclusion

Except for transient phenomena, the classical theory of circuits remains valid for fluctuations. As long as the purpose sought It is the calculation of the mean square values (of its voltage or current) and not the calculation of the voltages (currents) of the circuit.

The correlation matrix

- **Spectrum of self and cross powers**

$$z(t) = x(t) + y(t)$$

$$|Z|^2 = (X+Y)(X^* + Y^*) = |X|^2 + |Y|^2 + XY^* + X^*Y \quad (1.39)$$

$$S'_f(Z) = \lim_{\tau\to\infty} \frac{|Z|^2}{\tau} = S'_f(XX) + S'_f(YY) + S'_f(XY) + S'_f(YX) \quad (1.40)$$

$$S'_f(XX) = \lim_{\tau\to\infty} \frac{\overline{X^*X}}{\tau} \quad \text{si} \quad S'_f(YY) = \lim_{\tau\to\infty} \frac{\overline{Y^*Y}}{\tau} \quad (1.41)$$

$$S'_f(XY) = \lim_{\tau\to\infty} \frac{\overline{X^*Y}}{\tau} \quad \text{si} \quad S'_f(YX) = \lim_{\tau\to\infty} \frac{\overline{Y^*X}}{\tau} \quad (1.42)$$

Considering a random process z (t) resulting from the overlap of two stationary processes x (t) and y (t), we compute the Fourier transforms of these fluctuations (truncated at t = τ), which we note Z, X and Y.

The bilateral spectrum is defined by (1.40), where in Eq. (1.41) we have the self spectral densities of x (t) and y (t), and in Eq. (1.42) we have the cross spectral densities of x (t) and y (t).

The correlation matrix

- **Properties**

$$S'_f(Z) = S'_f(XX) + S'_f(YY) + 2\Re\left[S'_f(XY)\right] \quad (1.43a)$$

$$S_f(Z) = S_f(XX) + S_f(YY) + 2\Re\left[S_f(XY)\right] \quad (1.43b)$$

Properties

1) Self spectral densities are real quantities.
2) The cross spectral densities are complex quantities.
3) The real parts of the cross densities are even functions, while the imaginary parts are odd functions. Therefore, Eq. (1.40) becomes (1.43a) and, as the real part is an even function, this relation also holds for the unilateral spectra, (1.43b).

Wiener-Khintchine's Theorem 2

$$S_f(XX) = 2\int_{-\infty}^{+\infty} \overline{x(t)x(t+s)}\exp(j\omega s)\,ds \quad (1.44a)$$

$$S_f(YY) = 2\int_{-\infty}^{+\infty} \overline{y(t)y(t+s)}\exp(j\omega s)\,ds \quad (1.44b)$$

$$S_f(XY) = 2\int_{-\infty}^{+\infty} \overline{x(t)y(t+s)}\exp(j\omega s)\,ds \quad (1.44c)$$

We can generalize the Wiener-Khintchine theorem in the form (1.44a-c).

The correlation matrix

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} S_f(XX) & S_f(XY) \\ S_f(YX) & S_f(YY) \end{bmatrix} \quad (1.45)$$

Properties

Im (C11) = Im (C22) = 0

$C_{12} = C_{21}{}^*$

C_{11}> = 0 if C_{22}> = 0

$det\ \{C\}$ = C_11 C_22 - | C_12 | ^ 2 ≥0

The noise associated with two fluctuations, partially correlated, can be characterized by the correlation matrix, which is defined with the help of the spectral densities of self and cross powers, according to the frequency, Eq. (1.45).

Properties

Im (C11) = Im (C22) = 0

C12 = C21 *

C11> = 0 if C22> = 0

det {C} = C_11 C_22 - | C_12 | ^ 2 ≥0

Correlation coefficient (Kleckner approximation)

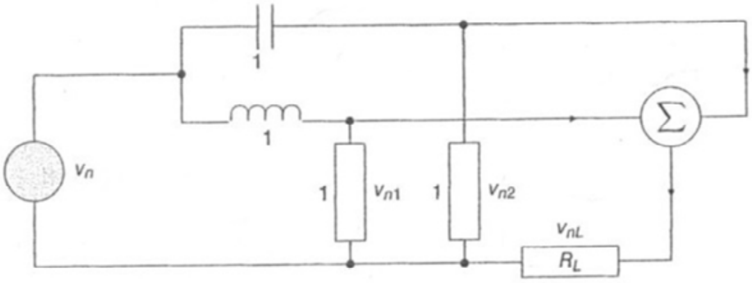
$$\Gamma_\omega(i,j) = \frac{S_\omega(i,j)}{\sqrt{S_\omega(i,i)S_\omega(j,j)}} \quad (1.46)$$

If we are interested in the noise of a multipole, whose equivalent generators of noise connected to the gates are described by the spectral density matrix, Kleckner proposes the following definitions for the correlation coefficients (which, in the frequency domain, are complex numbers), Eq. (1.46) .

Properties

1. $\Gamma_{-\omega}(i,j) = \Gamma_\omega^*(i,j) = \Gamma_\omega(j,i)$.
2. $\Gamma_\omega(i,j) \leq 1$, at all frequencies.
3. $|\Gamma_\omega(i,j)|$ is related to the total correlation between sources i and j.
4. The "effective" correlation (from the power density point of view) is specified by $\mathcal{R}\{\Gamma_\omega(i,j)\}$. If $\mathcal{R}\{\Gamma_\omega(i,j)\} = 0$, the interaction between the power spectral densities of sources i and j is null.

Example



$$2)\ \Gamma_\omega(1,2) = -j$$

Fig.1.7

$$v_{n1} = \frac{1}{1+j\omega} v_n \quad \text{deci} \quad |v_{n1}|^2 = \frac{1}{1+\omega^2} v_n^2$$

$$v_{n2} = \frac{1}{1+1/j\omega} v_n \quad \text{deci} \quad |v_{n2}|^2 = \frac{\omega^2}{1+\omega^2} v_n^2$$

$$v_{n1} = \frac{1}{1+j\omega} v_n \quad \text{deci} \quad |v_{n1}|^2 = \frac{1}{1+\omega^2} v_n^2$$

$$v_{n2} = \frac{1}{1+1/j\omega} v_n \quad \text{deci} \quad |v_{n2}|^2 = \frac{\omega^2}{1+\omega^2} v_n^2$$

$$S_\omega(1,1) = \overline{v_{n1} v_{n1}^*} = \overline{|v_{n1}|^2} = \frac{\overline{v_n^2}}{1+\omega^2} = \frac{K}{1+\omega^2}$$

$$S_\omega(2,2) = \overline{v_{n2} v_{n2}^*} = \overline{|v_{n2}|^2} = \frac{\omega^2}{1+\omega^2} \overline{v_n^2} = \frac{\omega^2}{1+\omega^2} K$$

$$S_\omega(1,2) = \overline{v_{n1} v_{n2}^*} = \overline{|v_{n2}|^2} = \overline{\frac{1}{1+j\omega} v_n \frac{1}{1-1/j\omega} v_n} = \frac{-j\omega}{1+\omega^2} K$$

Let the circuit in figure (1.7). where the noise generator is of the "white noise" type. Its power spectral density is denoted by K, the component values are normalized and the load is RL. Is required:

1. Spectral noise densities at the terminals of the resistors.
2. Correlation coefficients, according to Kleckner.

The solution

The interpretation is as follows: the two sources vn1 and vn2 (which are totally correlated because they come from the same generator vn) have a null interaction between their powers (because the real part of the correlation coefficient Γ_ω (1,2) is null).